Graded identities of block-triangular matrices *

Diogo Diniz Pereira da Silva e Silva †

Unidade Acadêmica de Matemática e Estatística
Universidade Federal de Campina Grande
Campina Grande, PB, Brazil

Thiago Castilho de Mello ‡

Instituto de Ciência e Tecnologia
Universidade Federal de São Paulo
São José dos Campos, SP, Brazil

September 15, 2015

Abstract

Let F be an infinite field and $UT(d_1,\ldots,d_n)$ be the algebra of upper block-triangular matrices over F. In this paper we describe a basis for the G-graded polynomial identities of $UT(d_1,\ldots,d_n)$, with an elementary grading induced by an n-tuple of elements of a group G such that the neutral component corresponds to the diagonal of $UT(d_1,\ldots,d_n)$. In particular, we prove that the monomial identities of such algebra follow from the ones of degree up to $2n-1$. Our results generalize for infinite fields of arbitrary characteristic, previous results in the literature which were obtained for fields of characteristic zero and for particular G-gradings. In the characteristic zero case we also generalize results for the algebra $UT(d_1,\ldots,d_n)\otimes C$ with a tensor product grading, where C is a color commutative algebra generating the variety of all color commutative algebras.

---

*partially supported by Fapesp grant No. 2014/10352-4, Fapesp grant No.2014/09310-5, CNPq grant No. 461820/2014-5, CNPq grant No. 480139/2012-1 and CAPES grant No. 99999.001558/2014-05

†diogo@dme.ufcg.edu.br

‡tcmello@unifesp.br

1 Introduction

Lef F be an infinite field and $UT(d_1, \dots, d_n)$ the algebra of upper block triangular matrices. It is the subalgebra of the matrix algebra $M_{d_1+\cdots+d_n}(F)$ consisting of the matrices

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ 0 & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{nn} \end{pmatrix},$$

where A_{ij} is a block of size $d_i \times d_j$. In this paper we study the graded polynomial identities of upper block triangular matrix algebras $UT(d_1, \dots, d_n)$ over an infinite field F. These algebras appear in the classification of minimal varieties (see[24]) and are generalizations of the matrix algebras (when $n = 1$) and the algebra $UT_n(F)$ of upper triangular matrices (when $d_1 = \cdots = d_n = 1$).

One of the main problems in the theory of PI-algebras is the (generalized) Specht problem about the existence, for a given class of algebras, of finite basis for the T-ideals of identities. This problem for the ordinary identites of associative algebras over a field of characteristic zero was solved by Kemer (see [26], [27]). In the case of associative algebras graded by a finite group it was solved by I. Sviridova [35] in the case of abelian groups and by E. Aljadeff and A. Kanel-Belov [1] in the general case. Over fields of positive characteristic however the situations is different and ideals of identities without finite basis exist (see for example [11], [25], [34]). The basis for the graded identities of $UT(d_1, \dots, d_n)$ in our main result (Theorem 3.7) is finite, provided that G is finite.

The algebras of block triangular matrices admit gradings by any group G in which the elementary matrices are homogeneous. These are called *elementary gradings* (or good gradings, see [10]). The algebras $UT_n(F)$ admit elementary gradings only (see [36]). Over an algebraically closed field of characteristic 0 every grading on $M_n(F)$ by a finite group is obtained by a certain tensor product construction from an elementary grading and a *fine* grading (see [9]). If moreover the group is abelian an analogous result holds for the algebra $UT(d_1, \dots, d_n)$ (see [37]).

Explicit basis for the identities are known for a few algebras only and for the algebras $UT(d_1, \dots, d_n)$ (over an infinite field) the only known basis for the ordinary identities are for the algebras $M_2(F)$ (see [32], [22], [28]) and $UT_n(F)$ (see [30]). In general, the ideal of identities of $UT(d_1, \dots, d_n)$ is

the product of the ideals of identities of the matrix algebras M_{d_i} (see [23]). An analogous property for the graded identities of block triangular matrix algebras was studied in [14]. Elementary gradings on $UT_n(F)$ and the corresponding graded identities were studied in [18] and in particular it was proved that elementary gradings can be distinguished by their graded identities. An analogous result for $UT(d_1, \ldots, d_n)$ with an elementary grading by an abelian group was obtained in [19].

When $char F = 0$ a complete description of the $\mathbb{Z}_2$-graded identities of $M_2(F)$ (and other PI-algebras) was given in [13]. Analogous basis for the identities of $M_n(F)$ with elementary $\mathbb{Z}$ and $\mathbb{Z}_n$ gradings were determined by Vasilovsky in [38], [39]. These results were also established for infinite fields (see [29], [4], [5]) and analogous results were obtained for related algebras (see [13],[15],[16]). Graded identities of $M_n(F)$ were studied more generally in [6] and in particular a basis for the graded identities of $M_n(F)$ with certain elementary gradings was determined. The elementary G-gradings considered are the ones induced by an n-tupple of pairwise different elements of G. The result considering an infinite field was obtained in [20]. In this case the basis is analogous to the one obtained by Vasilovsky and some monomial identities may be necessary. Recall that a G-grading on an algebra A is called *nondegenerate* if the ideal of graded identities of A contains no monomials. These types of gradings were studied in [2], [3]. Vasilovsky proved that $\mathbb{Z}_n$-grading on $M_n(F)$ is nondegenerate and that for the $\mathbb{Z}$-gradings one needs to consider the monomial identities of degree 1 corresponding to the homogeneous components of dimension 0.

In this paper we prove that the basis given in [6] holds for the algebras $UT(d_1, \ldots, d_n)$ over an infinite field F. Moreover we prove that it is only necessary to include the monomial identities of degree up to $2n-1$ in the basis. In [12] a similar result was proved for $\mathbb{Z}_n$-graded identities. In Section 4 we assume the field F has characteristic zero and generalize the result for the tensor product $UT(d_1, \ldots, d_n) \otimes E$, where E denotes the Grassmann algebra with its canonical $\mathbb{Z}_2$-grading, with the tensor product grading. Finally based on the results of [6] we generalize our main result to the tensor product $UT(d_1, \ldots, d_n) \otimes C$ where C is a color commutative algebra generating the variety of all color commutative algebras. We remark that C is a is a color commutative algebra generating the variety of all color commutative algebras if and only if it has a regular grading. Such gradings were recently studied in [2].

The ideas used in the present paper are similar to those in [4], [5], [6], [12], and [17]

2 Preliminaries

In this paper we consider associative algebras over an infinite field F and vector spaces are also considered over F.

2.1 Graded algebras and graded polynomial identities

Let A be an algebra and G a group. A G-*grading* on A is a vector space decomposition $A = \oplus_{g\in G} A_g$ compatible with the multiplication of the algebra in the sense that the inclusions

$$A_g A_h \subseteq A_{gh}$$

hold for any g and h in G. A nonzero element a in $\cup_{g\in G} A_g$ is called a *homogeneous element*. Clearly to every homogeneous element a corresponds an element g in G such that $a \in A_g$. We say that this g is the *degree of* a in the given G-grading. The set $\{g \in G | A_g \neq 0\}$ is the *support* of the grading and is denoted by *supp*A.

A subspace V of A is a *homogeneous subspace* if $V = \oplus_{g\in G}(V \cap A_g)$. A subalgebra B is a *homogeneous subalgebra* if it is homogeneous as a subspace and in this case $B = \oplus_{g\in G} B_g$, where $B_g = B \cap A_g$, is a G-grading on B. The G-grading on a homogeneous subalgebra B of A is assumed to be this one.

Let $X = \cup X_g$ be a disjoint union of a family of countable sets $X_g = \{x_g^{(1)}, x_g^{(2)}, \dots\}$ and $F\langle X|G\rangle$ be the free associative algebra, freely generated by X. If G is clear from the context, we denote it simply by $F\langle X\rangle$. A polynomial $f(x_{g_1}^{(1)}, \dots, x_{g_n}^{(n)})$ is a graded polynomial identity for $A = \oplus_{g\in G} A_g$ if we have $f(a_{g_1}^{(1)}, \dots, a_{g_n}^{(n)}) = 0$ for any $a_{g_1}^{(1)} \in A_{g_1}, \dots, a_{g_n}^{(n)} \in A_{g_n}$. The set $T_G(A)$ of all graded polynomial identities of A is an ideal of $F\langle X\rangle$ invariant under all graded endomorphisms of this algebra, i.e. it is a T_G-ideal. If S is a set of polynomials in $F\langle X\rangle$ the intersection U of all T_G-ideals containing S is a T_G-ideal. In this case, we say that S is a *basis* for U. Two sets are equivalent if they generate the same T_G-ideal. Since the field F is infinite it is well known that every polynomial f in $F\langle X\rangle$ is equivalent to a finite collection of multihomogeneous identities. Hence we may reduce our considerations to multihomogeneous polynomials.

2.2 Elementary gradings on block-triangular matrices

Let $(g_1, \dots, g_m)$ be an m-tuple of elements of G and $A = M_m(F)$ be the full matrix algebra of order m. If we set A_g to be the subspace spanned by the

elementary matrices e_{ij} such that $g_i^{-1}g_j = g$ then we have

$$A = \oplus_{g\in G} A_g$$

and this decomposition is a G-grading. Let B be a subalgebra of A generated by elementary matrices. Then B is a homogeneous subalgebra. In particular $UT(d_1, \ldots, d_n)$ is a homogeneous subalgebra of $M_m(F)$, where $d_1+\cdots+d_n = m$. We say that the G-grading on $UT(d_1, \ldots, d_m)$ (and more generally on B) is the elementary grading induced by $(g_1, \ldots, g_n)$.

Let e denote the unit of the group G and consider $B = UT(d_1, \ldots, d_m)$ with the elementary grading induced by $(g_1, \ldots, g_n)$. The elementary matrices e_{ii} have degree e and therefore the dimension of the component B_e is $\geq n$. We have $dim B_e = n$ if and only if the elements in the n-tuple inducing the grading are pairwise distinct. Equivalently the polynomial $x_e^{(1)}x_e^{(2)} - x_e^{(2)}x_e^{(1)}$ is a graded identity for B.

2.3 Generic Graded Algebras

Let $\mathbf{g} = (g_1, \ldots, g_n)$ be a n-tuple of pairwise distinct elements of G. Denote by $A = \oplus_{g\in G} A_g$ the algebra $M_n(F)$ with the elementary grading induced by $\mathbf{g}$. Let B be a subalgebra of A with basis $\{e_{i_1j_1}, \ldots, e_{i_lj_l}\}$ as a vector space. Denote by G_0 (resp. G_0^A) the support of the grading on B (resp. A).

Let g be an element in the support G_0 of the grading of B. Denote by $D_{\widehat{g}}$ the set of indexes $i \in \{i_1, \ldots, i_l\}$ such that for some $j \in \{j_1, \ldots, j_l\}$ the matrix unit e_{ij} has degree g. Recall that the n-tuple $\mathbf{g}$ consists of pairwise distinct elements of G. This implies that for each $i \in D_{\widehat{g}}$ there exists exactly one index in $\{j_1, \ldots, j_l\}$, denoted by $\widehat{g}(i)$, such that $e_{i\widehat{g}(i)} \in B_g$. Thus we obtain a function $\widehat{g} : D_{\widehat{g}} \to \{j_1, \ldots, j_l\}$ for each $g \in G_0$. With this notation $\{e_{i\widehat{g}(i)} | i \in D_{\widehat{g}}\}$ is a basis for B_g.

Denote by Ω the algebra of polynomials in commuting variables

$$\Omega = F[\xi_{ij}^{(k)} \mid i, j = 1, 2, \ldots, n;\ k = 1, 2, \ldots].$$

The algebra $M_n(\Omega)$ has a natural G-grading where the homogeneous component of degree g is the subspace generated by the matrices $m_{ij}e_{ij}$, where $e_{ij} \in A_g$ and m_{ij} is a monomial in Ω.

Definition 2.1 *For each $g \in G_0$ and each natural number k the element*

$$\xi_g^{(k)} = \sum_{i\in D_{\widehat{g}}} \xi_{i\widehat{g}(i)}^{(k)} e_{i\widehat{g}(i)}$$

of $M_n(\Omega)$ is called a ***graded generic element****. The algebra $G(B)$ generated by the $\xi_g^{(k)}$, $g \in G_0$, $k = 1, 2, \ldots$ is called the* ***algebra of graded generic elements of*** *B.*

The algebra $G(B)$ is a homogeneous subalgebra of $M_n(\Omega)$ and is a graded algebra with the inherited grading. If $B = A$ the above construction yields the graded algebra $G(A)$ of generic elements of A. The generic element in $G(A)$ corresponding to $g \in G_0^A$ and k will be denoted by $\xi_g^{(k,A)}$. The following result is well known.

Theorem 2.2 *Let F be an infinite field. The algebra $G(B)$ is isomorphic as a graded algebra to the relatively free G-graded algebra $F\langle X\rangle / Id_G(B)$.*

Proof. The homomorphism $\Theta : F\langle X\rangle \to G(B)$ induced by mapping $x_g^{(i)} \mapsto \xi_g^{(i)}$ is clearly onto. Moreover as in the case of the generic matrix algebra (see [24, Theorem 1.4.4]) we have $ker\Theta = T_G(B)$ and the result follows. □

3 The main result

Given $g_1, g_2, \ldots, g_p \in G_0$ we consider the composition $\nu = \widehat{g_p} \cdots \widehat{g_1}$ of the corresponding functions. This may not be well defined and we will prove in the next lemma that in this case the monomial $x_{g_1}^{(1)} \cdots x_{g_p}^{(p)}$ is a graded identity for B. Otherwise its domain $D_\nu = D_{\widehat{g_p} \cdots \widehat{g_1}}$ is the set of $i \in \{i_1, \ldots, i_l\}$ for which the image $\widehat{g_p}(\ldots(\widehat{g_1}(i))\ldots)$ is well defined. In this case $\{e_{i\nu(i)} | i \in D_\nu\}$ is a basis for the subspace spanned by $B_{g_1} \cdots B_{g_p}$.

Lemma 3.1 *Let $h_1, h_2, \ldots, h_p$ be elements in G_0. If $D_{\widehat{h_p} \cdots \widehat{h_1}} = \emptyset$ then $\xi_{h_1}^{(i_1)} \xi_{h_2}^{(i_2)} \cdots \xi_{h_p}^{(i_p)} = 0$. Moreover if the set $D_{\widehat{h_p} \cdots \widehat{h_1}}$ is nonempty then the i-th line of the matrix $\xi_{h_1}^{(i_1)} \xi_{h_2}^{(i_2)} \cdots \xi_{h_p}^{(i_p)}$ is nonzero if and only if $i \in D_{\widehat{h_p} \cdots \widehat{h_1}}$. In this case if $j = \widehat{h_p} \cdots \widehat{h_1}(i)$, the only nonzero entry in the i-th line is a monomial of Ω in the j-th column.*

Proof. The proof is by induction on the length p of the product. The result for $p = 1$ follows directly from Definition 2.1. Hence we consider $p > 1$ and assume the result for products of length $p - 1$. Let us consider first the case $D_{\widehat{h_p} \cdots \widehat{h_1}} \neq \emptyset$. In this case $D_{\widehat{h_{p-1}} \cdots \widehat{h_1}} \neq \emptyset$ and we denote $\nu = \widehat{h_{p-1}} \cdots \widehat{h_1}$. The induction hypothesis implies that there exists monomials m_i, where $i \in D_{\widehat{h_{p-1}} \cdots \widehat{h_1}}$, such that

$$\xi_{h_1}^{(i_1)}\xi_{h_2}^{(i_2)}\cdots\xi_{h_p}^{(i_p)} = \left(\sum_{i\in D_{\widehat{h_{p-1}}\cdots\widehat{h_1}}} m_i e_{i\nu(i)}\right)\left(\sum_{j\in D_{\widehat{h_p}}} \xi_{j\widehat{h_p}(j)}^{(i_p)} e_{j\widehat{h_p}(j)}\right). \qquad (1)$$

Note that $e_{i\nu(i)}e_{j\widehat{h_p}(j)} \neq 0$ for some j if and only if $i \in D_{\widehat{h_p}\cdots\widehat{h_1}}$ and in this case the product equals $e_{i\widehat{h_p}(j)}$. Hence we obtain

$$\xi_{h_1}^{(i_1)}\xi_{h_2}^{(i_2)}\cdots\xi_{h_p}^{(i_p)} = \sum_{i\in D_{\widehat{h_p}\cdots\widehat{h_1}}} (m_i\xi_{\nu(i)\widehat{h_p}(\nu(i))}^{(i_p)})e_{i\widehat{h_p}(\nu(i))},$$

and the result follows. Now assume that $D_{\widehat{h_p}\cdots\widehat{h_1}} = \emptyset$. If $D_{\widehat{h_{p-1}}\cdots\widehat{h_1}} = \emptyset$ then by the induction hypothesis $\xi_{h_1}^{(i_1)}\xi_{h_2}^{(i_2)}\cdots\xi_{h_{p-1}}^{(i_{p-1})} = 0$ and the result holds. Moreover if $D_{\widehat{h_{p-1}}\cdots\widehat{h_1}} \neq \emptyset$ then we may write the product $\xi_{h_1}^{(i_1)}\xi_{h_2}^{(i_2)}\cdots\xi_{h_p}^{(i_p)}$ as in (1). Since $D_{\widehat{h_p}\cdots\widehat{h_1}} = \emptyset$ every product $e_{i\nu(i)}e_{j\widehat{h_p}(j)}$ equals zero and therefore $\xi_{h_1}^{(i_1)}\xi_{h_2}^{(i_2)}\cdots\xi_{h_p}^{(i_p)} = 0$. □

Corollary 3.2 *If a monomial $x_{h_1}^{(i_1)}\ldots x_{h_p}^{(i_p)}$ in $F\langle X\rangle$ is a graded identity for B then it is a consequence of a monomial in $T_G(B)$ of length at most $2n-1$.*

Proof. The result follows if we prove that every monomial in $T_G(B)$ of length $p > 2n-1$ is a consequence of a monomial identity of length at most $p-1$. Let $m = x_{h_1}^{(i_1)}\ldots x_{h_p}^{(i_p)}$ be a monomial identity for B. Clearly we may assume that $h_i \in G_0$, $i = 1, 2, \ldots, p$. If $D_{\widehat{h_r}\widehat{h_{r-1}}\ldots\widehat{h_1}} = \emptyset$ for some $r < p$ then Lema 3.1 implies that $\xi_{h_1}^{(i_1)}\ldots\xi_{h_r}^{(i_r)} = 0$. Hence $x_{h_1}^{(i_1)}\ldots x_{h_p}^{(i_p)}$ is an identity for B and m is a consequence of this monomial. Thus we assume that $D_r = D_{\widehat{h_r}\widehat{h_{r-1}}\ldots\widehat{h_1}}$ is nonempty for $r < p$ and denote by I_r the image of the composition $\widehat{h_r}\widehat{h_{r-1}}\ldots\widehat{h_1}$. Notice that

$$D_1 \supseteq D_2 \supseteq \cdots \supseteq D_{p-1} \supseteq D_p = \emptyset. \qquad (2)$$

Assume that there exists r such that $D_r = D_{r+1} = D_{r+2}$. The equality $D_r = D_{r+2}$ implies that $I_r \subseteq D_{\widehat{h_{r+2}}\widehat{h_{r+1}}}$. Clearly $D_{\widehat{h_{r+2}}\widehat{h_{r+1}}} \subseteq D_{\widehat{h}}$, where $h = h_{r+1}h_{r+2}$. Therefore $I_r \subseteq D_{\widehat{h}}$ and we conclude that $D_{\widehat{h}\widehat{h_r}\widehat{h_{r-1}}\ldots\widehat{h_1}} = D_r$. Since $D_r = D_{r+2}$ this implies that the compositions $\widehat{h}\widehat{h_r}\widehat{h_{r-1}}\ldots\widehat{h_1}$ and $\widehat{h_{r+2}}\widehat{h_{r+1}}\widehat{h_r}\ldots\widehat{h_1}$ have the same domain. Moreover the equality in G,

$h_1h_2\cdots h_rh = h_1h_2\cdots h_{r+1}h_{r+2}$ implies that for every $i \in D_{r+2}$, $\widehat{h}\widehat{h_r}\dots\widehat{h_1}(i) = \widehat{h_{r+2}}\widehat{h_{r+1}}\widehat{h_r}\dots\widehat{h_1}(i)$. Hence $\widehat{h}\widehat{h_r}\dots\widehat{h_1} = \widehat{h_{r+2}}\widehat{h_{r+1}}\widehat{h_r}\dots\widehat{h_1}$ and therefore we have $D_{\widehat{h_p}\dots\widehat{h_{r+3}}\widehat{h}\widehat{h_r}\dots\widehat{h_1}} = D_p = \emptyset$. It follows from Lemma 3.1 that the monomial $m' = x_{h_1}^{(i_1)}\dots x_{h_r}^{(i_r)}(x_h^{(i_{r+1})})x_{h_{r+3}}^{(i_{r+3})}\dots x_{h_p}^{(i_p)}$, where $i_{r+1} \notin \{i_1,\dots,i_r\}$, is an identity for B. Clearly m is a consequence of m'. It remains only to verify that if $p > 2n-1$ there exists r such that $D_r = D_{r+1} = D_{r+2}$. First notice that if $|D_1| = n$ then $\{i_1,\dots,i_l\} = \{1,2,\dots,n\} = \{j_1,\dots,j_l\}$ and $\widehat{h_1}$ is a bijection in this set. Therefore $D_p = \emptyset$ implies that $D_{\widehat{h_p}\widehat{h_{p-1}}\dots\widehat{h_2}} = \emptyset$. By Lemma 3.1 the monomial $x_{h_2}^{(i_2)}x_{h_3}^{(i_3)}\dots x_{h_p}^{(i_p)}$ is an identity and clearly m is a consequence of it. Therefore we may assume now that $|D_1| \le n-1$. In this case there are at most $n-1$ proper inclusions in (2) and if $p > 2n-1$ there are two consecutive equalities, i. e., there exists r such that $D_r = D_{r+1} = D_{r+2}$. □

We consider the following graded polynomials:

$$x_e^{(1)}x_e^{(2)} - x_e^{(2)}x_e^{(1)}, \text{ if } e \in G_0 \tag{3}$$

$$x_g^{(1)}x_{g^{-1}}^{(2)}x_g^{(3)} - x_g^{(3)}x_{g^{-1}}^{(2)}x_g^{(1)} \text{ if } e \neq g \text{ and } B_g \neq 0 \tag{4}$$

$$x_g^{(1)} \text{ if } B_g = 0. \tag{5}$$

Lemma 3.3 *The algebra B with the elementary grading induced by an n-tuple $g = (g_1,\dots,g_n)$ of pairwise distinct elements of G satisfies the graded polynomial identities* (3) − (5).

Proof. Clearly the polynomials in (5) are identities for B. Since the elements in $\mathbf{g} = (g_1,\dots,g_n)$ are pairwise different if $e \in G_0$ the graded generic matrices $\xi_e^{(i)}$ are diagonal. Hence we have the graded identity (3). Since (4) is multilinear, in order to verify that it is a graded identity substitute $x_g^{(1)}, x_g^{(3)}$ by $e_{ij}, e_{kl} \in B_g$ respectively and $x_{g^{-1}}^{(2)}$ by $e_{rs} \in B_{g^{-1}}$. If $(e_{ij}e_{rs}e_{kl}) \neq 0$ then $j = r$ and $s = k$. Moreover e_{is} and e_{rl} are in A_e and therefore $i = s$ and $r = l$. Hence in this case $e_{ij} = e_{kl}$ and the result of the substitution is zero. Analogously if $(e_{kl}e_{rs}e_{ij}) \neq 0$ the result is zero. The remaining case to consider is $(e_{ij}e_{rs}e_{kl}) = 0 = (e_{kl}e_{rs}e_{ij})$ and the result is also 0. □

Proposition 3.4 *[20, Lemma 4.6] Let U_A denote the T_G-ideal generated by the identities* (3) − (5) *satisfied by the matrix algebra A and let $\xi_g^{(i,A)}$, $g \in G_0^A$, $i = 1,2,\dots$ denote the generic elements in $G(A)$. If the monomials $m(x_{h_1}^{(1)},\dots,x_{h_p}^{(p)})$ and $n(x_{h_1}^{(1)},\dots,x_{h_p}^{(p)})$ in $F\langle X\rangle$ are such that the matrices*

$n(\xi_{h_1}^{(1,A)} \ldots, \xi_{h_p}^{(p,A)})$ *and* $n(\xi_{h_1}^{(1,A)} \ldots, \xi_{h_p}^{(p,A)})$ *have the same position the same non-zero entry then*

$$m(x_{h_1}^{(1)}, \ldots, x_{h_p}^{(p)}) \equiv n(x_{h_1}^{(1)}, \ldots, x_{h_p}^{(p)}) \text{ modulo } U_A.$$

Next we generalize this proposition to the case of a subalgebra B of $A = M_n(F)$ generated by elementary matrices. Note that the algebra $G(B)$ is a homomorphic image of the algebra $G(A)$ by Theorem 2.2. The homomorphism constructed in the following remark will be usefull.

Remark 3.5 *We construct a homomorphism from* $G(A)$ *to* $G(B)$ *as follows: the map* $x_{ij}^{(k)} \mapsto \chi_{ij} x_{ij}^{k}$ *where* $\chi_{ij} = 1$ *if* $e_{ij} \in B_g$ *and* $\chi_{ij} = 0$ *if* $e_{ij} \notin B_g$ *induces an endomorphism* θ *of* Ω *extending this map. Hence* $\Theta : M_n(\Omega) \to M_n(\Omega)$ *given by* $\Theta(\sum p_{ij} e_{ij}) = \sum \theta(p_{ij}) e_{ij}$ *is an endomorphism of* $M_n(\Omega)$*. From the definiton of* θ *if follows that* $\Theta(\xi_{ij}^{(k,A)}) = \xi_{ij}^{(k)}$ *and therefore* $\Theta(G(A)) = G(B)$*. The restriction to* $G(A)$ *gives the desired homomorphism (also denoted by* Θ*).*

Corollary 3.6 *Let* B *be a subalgebra of* $M_n(F)$ *generated by elementary matrices with the induced* G*-grading and* U_0 *be the* T_G*-ideal generated by the identities* $(3) - (5)$ *satisfied by the graded algebra* B*. If* $m(x_{h_1}^{(1)}, \ldots, x_{h_p}^{(p)})$ *and* $n(x_{h_1}^{(1)}, \ldots, x_{h_p}^{(p)})$ *are two monomials in* $F\langle X \rangle$ *such that the matrices* $m(\xi_{h_1}^{(1)}, \cdots, \xi_{h_p}^{(p)})$ *and* $n(\xi_{h_1}^{(1)}, \cdots, \xi_{h_p}^{(p)})$ *have in the same position the same nonzero entry then*

$$m(x_{h_1}^{(1)}, \ldots, x_{h_p}^{(p)}) \equiv n(x_{h_1}^{(1)}, \ldots, x_{h_p}^{(p)}) \text{ modulo } U_0.$$

Proof. Let $\tilde{m}(x_{g_1}^{(1)} \cdots x_{g_n}^{(n)})$ be a monomial in $F\langle X \rangle$. Let Θ be the homomorphism constructed in the previous remark. We have

$$\Theta(\tilde{m}(\xi_{g_1}^{(i_1,A)} \cdots \xi_{g_1}^{(i_n,A)})) = \tilde{m}(\xi_{g_1}^{(i_1)} \cdots \xi_{g_n}^{(i_n)}). \tag{6}$$

It follows from Lemma 3.1 that the entries of $\tilde{m}(\xi_{g_1}^{(i_1,A)} \cdots \xi_{g_1}^{(i_1,A)})$ are monomials in Ω. Note that if p is a monomial in Ω then $\theta(p)$ is either 0 or p. Hence (6) implies that the nonzero entries of $\tilde{m}(\xi_{g_1}^{(i_1)} \cdots \xi_{g_n}^{(i_n)})$ equal the corresponding entries of $\tilde{m}(\xi_{g_1}^{(i_1,A)} \cdots \xi_{g_n}^{(i_n,A)})$. Thus the matrices $m(\xi_{h_1}^{(1,A)}, \cdots, \xi_{h_p}^{(p,A)})$ and $n(\xi_{h_1}^{(1,A)}, \cdots, \xi_{h_p}^{(p,A)})$ have in the same position the same nonzero entry. Therefore Proposition 3.4 imples that $m \equiv n$ modulo U_A. The result is then a consequence of the inclusion $U_A \subseteq U_0$. To prove this we verify that every generator of U_A is in U_0 and this follows from the inclusion $G_0 \subset G_0^A$. $\square$

Theorem 3.7 *Let G be a group and let $\boldsymbol{g} = (g_1, \ldots, g_n) \in G^n$ induce an elementary G-grading of $M_n(F)$, where the elements $g_1, \ldots, g_n$ are pairwise different. If B is a subalgebra of $M_n(F)$ generated by elementary matrices e_{ij} then a basis of the graded polynomial identities of B consists of* $(3)-(5)$ *and a finite number of identities of the form $x_{h_1 1} \ldots x_{h_p p}$, where $2 \leq p \leq 2n-1$.*

Proof. Let U be the T_G-ideal of $F\langle X\rangle$ generated by the polynomials (3) – (5) together with the monomial identities $x^{(1)}_{h_1} \ldots x^{(p)}_{h_p}$ of B with $2 \leq p \leq 2n-1$. It follows from Lemma 3.3 that $U \subseteq T_G(B)$. Hence to prove the theorem it is enough to show that every multihomogeneous G-graded identity of B lies in U. Assume, on the contrary, that f is a multihomogeneous graded identity that does not lie in U. We write $f \equiv \sum_{i=1}^{k} \alpha_i m_i$ modulo U, where the α_i are non-zero scalars and m_i are monomials in $F\langle X\rangle$. We may assume that the number k of nonzero coefficients is minimal. If $k = 1$ then m_1 is an identity for B and Corollary 3.2 implies that it lies in U which is a contradiction. We now consider $k > 1$. Denote by $\overline{m_i}$ the matrix in $G(B)$ that is the result of substituting every variable $x^{(j)}_g$ in m_i for the corresponding generic matrix $\xi^{(i)}_g$. By the minimality of k the monomials m_i are not identities for B and in particular $\overline{m_1}$ has a nonzero entry. Moreover we have

$$-\alpha_1 \overline{m_1} = \sum_{i=2}^{k} \alpha_i \overline{m_i}.$$

It follows from Lema 3.1 that the nonzero entries of the matrices $\overline{m_i}$ are monomials in Ω. Therefore there exists a $j > 1$ such that $\overline{m_j}$ and $\overline{m_1}$ have in the same position the same nonzero entry. Thus Corollary 3.6 implies that $m_1 \equiv m_j$ modulo U. Hence $f \equiv (\alpha_1 + \alpha_j)m_1 + \sum_{i\neq j} m_i$ modulo U. This last polynomial is an identity for B that does not lie in U with fewer nonzero coefficients than f and this is a contradiction. □

Recall that a G-grading on an algebra A is called nondegenerate if for every integer r and any tuple $(g_1, \ldots, g_r) \in G^r$ the monomial $x^{(1)}_{g_1} \cdots x^{(r)}_{g_r}$ is not a graded identity for A (see [2, Observation 2.2]). A stroger condition is that $A_g A_h = A_{gh}$ for every $g, h \in G$ and in this case the grading is called *strong*. The $\mathbb{Z}_n$-grading considered by Vasilovsky in [39] is strong and in particular nondegenerate and the basis determined consists of (3) and (4). In the next corollary we consider elementary G-gradings on $M_n(F)$ that are closely related to this grading.

Corollary 3.8 *Let G be a finite group with unit e and let $M_n(F)$ be endowed with an elementary grading such that $x^{(1)}_e x^{(2)}_e - x^{(2)}_e x^{(1)}_e$ is a graded*

polynomial identity. If the G-grading is nondegenerate then a basis for the graded identities of $M_n(F)$ consists of the polynomials (3) *and* (4). *Moreover in this case the grading is strong and G is a group of order n.*

Proof. Let $\mathbf{g} = (g_1, \ldots, g_n) \in G^n$ be a tuple inducing the elementary grading. If $g_i = g_j$ for some $i \neq j$ then the elementary matrices e_{ij} and e_{ji} have degree e and $e_{ij}e_{ji} - e_{ji}e_{ij} \neq 0$. Hence $x_e^{(1)}x_e^{(2)} - x_e^{(2)}x_e^{(1)}$ is not a graded identity and this is a contradiction. Thus the elements in the tuple $\mathbf{g}$ are pairwise different. Since the grading is nondegenerate it follows from Theorem 3.7 that the polynomials (3) and (4) are a basis for the graded identities. Now we prove the last assertion. Note that since $\mathbf{g}$ consists of pairwise different elements we have $|G| \geq n$. We claim that if $|G| > n$ then there exists $g_1, \ldots, g_n \in G$ such that $x_{g_1}^{(1)} \cdots x_{g_n}^{(n)}$ is a graded identity. Clearly it follows from this claim that $|G| = n$. We construct the sequence as follows: since $|G| > n$ we let $g_1 \in G$ such that none of the elementary matrices $e_{11}, e_{12}, \ldots, e_{1n}$ have degree g_1. Clearly the first line of $\xi_{g_1}^{(1)}$ is zero. Then we choose g_2 such that the second line of $\xi_{g_1g_2}^{(1)}$ is zero. Inductively we choose g_i such that the i-th line of ξ_g^1 is zero, where $g = g_1 \cdots g_i$. Note that the first line of $\xi_{g_1}^{(1)}\xi_{g_2}^{(2)}$ is zero since the first line of $\xi_{g_1}^{(1)}$ is zero. Moreover the second line of $\xi_{g_1}^{(1)}\xi_{g_2}^{(2)}$ is also zero because the second line of $\xi_{g_1g_2}^{(1)}$ is zero. It follows by induction that the first i lines of $\xi_{g_1}^{(1)} \cdots \xi_{g_i}^{(i)}$ are zero. Hence $\xi_{g_1}^{(1)} \cdots \xi_{g_n}^{(n)} = 0$ and it follows from Lemma 3.1 that $x_{g_1}^{(1)} \cdots x_{g_n}^{(n)}$ is a graded identity for $M_n(F)$. Now we prove that the grading is strong. Let $g \in G$. Note that for each i the elementary matrices $e_{i1}, \ldots, e_{in}$ have pairwise different degrees and since $|G| = n$ the sequence of degrees is just a reordering of the elements of G. Thus there exists j such that e_{ij} has degree g. Hence we obtain $A_gA_h = A_{gh}$ for any $g, h \in G$. $\square$

Remark 3.9 *The proof of the last assertion in the previous lemma is based on the proof of Lemma 3.3 in [2]. In this lemma a characterization of nondegenerate gradings on finite dimensional G-simple algebras is given.*

Corollary 3.10 *Let G be a group. If $UT(d_1, \ldots, d_n)$ has an elementary grading such that the polynomials* (3) *and* (4) *are a basis for the graded polynomial identities of this graded algebra then $n = 1$,i.e., $UT(d_1, \ldots, d_n) = M_{d_1}(F)$.*

Proof. If $n > 1$ then we apply the previous lemma to each block A_{ii} to obtain a monomial that is a graded identity for M_{d_i}. The product of copies of these

monomials in disjoint sets of variables is a monomial m such that the result of any substitution lies in the jacobson radical J of $UT(d_1, \ldots, d_n)$. Since J is a nilpotent ideal, say $J^k = 0$, the product of k copies of m in disjoint sets of variables is a monomial identity. $\square$

4 Matrices over the Grassmann algebra

We now turn our attention to matrices over the Grassmann algebra.

In this section we suppose F is a field of characteristic zero and we denote by E the Grassmann algebra of an infinite dimensional vector space over F with its natural $\mathbb{Z}_2$-grading $E = E_0 \oplus E_1$ induced by the length of its monomials. For more information concerning the Grassmann algebra, see [21] .

We use the results of the previous sections and results of [17] to find a basis for the $G \times \mathbb{Z}_2$-graded polynomial identities of $UT(d_1, \ldots, d_n; E)$: the algebra of block-triangular matrices over the Grassmann algebra, which is isomorphic to the tensor product $UT(d_1, \ldots, d_n) \otimes E$, and more generally of the algebra $B \otimes E$, where B is a G-graded subalgebra of $M_n(F)$ generated by elementary matrices with an elementary grading induced by an n-tuple $(g_1, \ldots, g_n)$ of pairwise distinct elements of G.

If B is a G-graded algebra the algebra $B \otimes E$ has a natural $G \times \mathbb{Z}_2$-grading induced by the gradings of B and of E. In such grading, the homogeneous component of degree (g, δ) is $(B \otimes E)_{(g,\delta)} = B_g \otimes E_\delta$.

In order to work with the $G \times \mathbb{Z}_2$-graded identities of $B \otimes E$, we now consider the free associative algebra $F\langle Z \rangle$, with $Z = X' \cup Y'$, where $X' = \cup X'_g$ is the set of graded variables with $G \times \mathbb{Z}_2$-degree $(g, 0)$ and $Y' = \cup Y'_g$ is the set of graded variables with $G \times \mathbb{Z}_2$-degree $(g, 1)$. We denote elements of X'_g and Y'_g respectively by $x_g^{(i)}$ and $y_g^{(i)}$, for $i \in \mathbb{N}$ and $g \in G$. From now on, the variables labeled as $z_g^{(i)}$ may be $x_g^{(i)}$ or $y_g^{(i)}$.

Recall that in [17] the authors define a map ζ_J, for $J \subseteq \mathbb{N}$, which maps multilinear identities of B into identities of $B \otimes E$. Such map is defined as follows.

First, we observe that $F\langle Z \rangle$ is both a $\mathbb{Z}_2$-graded algebra and G-graded algebra. Concerning the $\mathbb{Z}_2$-grading of $F\langle Z \rangle$, one defines the map ζ as follows. If m is a multilinear monomial let $i_1 < \cdots < i_k$ be the indices with odd $\mathbb{Z}_2$-degree occurring in m. Then for some $\sigma \in \mathrm{Sym}(\{\mathrm{i}_1, \ldots, \mathrm{i}_k\})$, we write

$$m = m_0 y_{g_1 \sigma(i_1)} m_1 \cdots y_{g_k \sigma(i_k)} m_{k+1}$$

where $m_0, \dots, m_{k+1}$ are monomials on even variables only. Then we define

$$\zeta(m) = (-1)^\sigma m$$

Definition 4.1 *Let $J \subseteq \mathbb{N}$. We define $\varphi_J : F\langle X\rangle \longrightarrow F\langle Z\rangle$ to be the unique G-homomorphism of algebras defined by*

$$\varphi_J(x_g^{(i)}) = \begin{cases} x_g^{(i)} & \text{if } i \notin J \\ y_g^{(i)} & \text{if } i \in J \end{cases}$$

Also for a multilinear monomial m we define $\zeta_J(m) = \zeta(\varphi_J(m))$

The map ζ_J extends by linearity to the space of all multilinear polynomials in $F\langle X\rangle$ and for each multilinear polynomial in $F\langle X\rangle$, $\zeta_J(f)$ is also a multilinear polynomial in $F\langle Z\rangle$.

We now recall Theorem 11 of [17].

Theorem 4.2 *Let A be a G-graded algebra and $\mathcal{E} \subset F\langle X|G\rangle$ be a system of multilinear generators for $T_G(A)$. Then the set*

$$\{\zeta_J(f) \,|\, f \in \mathcal{E},\ J \subseteq \mathbb{N}\}$$

is system of multilinear generators of $T_{G\times\mathbb{Z}_2}(A \otimes E)$

Since the basis of the graded polynomial identities of $UT(d_1, \dots, d_m)$, described in Theorem 3.7 contains polynomials in at most $2n-1$ variables, it is enough to consider $J \subset \{1, \dots, 2n-1\}$.

Lemma 4.3 *Applying the map ζ_J to the polynomial $x_e^{(1)}x_e^{(2)} - x_e^{(2)}x_e^{(1)}$ we obtain up to endomorphisms of $F\langle X|G\times\mathbb{Z}_2\rangle$ the following polynomials*

$$x_e^{(1)}z_e^{(2)} - z_e^{(2)}x_e^{(1)} \tag{7}$$

$$y_e^{(1)}y_e^{(2)} + y_e^{(2)}y_e^{(1)} \tag{8}$$

Applying the map ζ_J (for $J \subseteq \{1,2,3\}$) to the polynomial $x_g^{(1)}x_{g^{-1}}^{(2)}x_g^{(3)} - x_g^{(3)}x_{g^{-1}}^{(2)}x_g^{(1)}$ we obtain up to endomorphisms of $F\langle X|G\times\mathbb{Z}_2\rangle$ the polynomials

$$z_g^{(1)}x_{g^{-1}}^{(2)}x_g^{(3)} - x_g^{(3)}x_{g^{-1}}^{(2)}z_g^{(1)} \tag{9}$$

$$x_g^{(1)}y_{g^{-1}}^{(2)}x_g^{(3)} - x_g^{(3)}y_{g^{-1}}^{(2)}x_g^{(1)} \tag{10}$$

$$y_g^{(1)}y_{g^{-1}}^{(2)}x_g^{(3)} + x_g^{(3)}y_{g^{-1}}^{(2)}y_g^{(1)} \tag{11}$$

$$y_g^{(1)}x_{g^{-1}}^{(2)}y_g^{(3)} + y_g^{(3)}x_{g^{-1}}^{(2)}y_g^{(1)} \tag{12}$$

$$y_g^{(1)}y_{g^{-1}}^{(2)}y_g^{(3)} + y_g^{(3)}y_{g^{-1}}^{(2)}y_g^{(1)} \tag{13}$$

Finally, if $m = x_{g_1}^{(1)} \cdots x_{g_p}^{(p)}$ is a G-graded monomial identity of the algebra B, generated by elementary matrices, for some $1 \leq p \leq 2n-1$ then up to some endomorphism of $F\langle X|G \times \mathbb{Z}_2\rangle$, $\zeta_J(m) = z_g^{(1)} \cdots z_g^{(p)}$

Proof. The proof consist of several applications of the map ζ_J for $J \subseteq \mathbb{N}$. For $f_1 = x_e^{(1)}x_e^{(2)} - x_e^{(2)}x_e^{(1)}$, it is enough to consider $J \subseteq \{1, 2\}$. So consider $J = \{1\}$ and $J = \{2\}$. Then $\zeta_{\{1\}}(f_1) = y_e^{(1)}x_e^{(2)} - x_e^{(2)}y_e^{(1)}$ and $\zeta_{\{2\}}(f_1) = -(y_e^{(2)}x_e^{(1)} - x_e^{(1)}y_e^{(2)})$, and the latter is the image of the former, by the endomorphism of $F\langle Z\rangle$, which permutes the indexes 1 and 2 of the variables and multiply the result by -1. For this reason, up to an endomorphism of $F\langle Z\rangle$, the image of f_1 is $x_e^{(1)}z_e^{(2)} - z_e^{(2)}x_e^{(1)}$, for $|J| \leq 1$. If $J = \{1, 2\}$, one obtains $\zeta_J = y_e^{(1)}y_e^{(2)} + y_e^{(2)}y_e^{(1)}$.

Similarly, one obtains the images by ζ_J of the polynomials

$$f_2 = x_g^{(1)}x_{g^{-1}}^{(2)}x_g^{(3)} - x_g^{(3)}x_{g^{-1}}^{(2)}x_g^{(1)} \text{ and } m = x_{g_1}^{(1)} \cdots x_{g_p}^{(p)}$$

□

By applying the above lemma and theorem we obtain:

Corollary 4.4 *Let F be a field of characteristic zero, G be a group and let $\boldsymbol{g} = (g_1, \ldots, g_n) \in G^n$ induce an elementary G-grading of $M_n(F)$, where the elements $g_1, \ldots, g_n$ are pairwise different. If B is a subalgebra of $M_n(F)$ generated by matrix units e_{ij}, then a basis of the graded polynomial identities of the algebra $B \otimes E$ consists of the polynomials (7) − (13) and a finite number of identities of the form $z_{g_1}^{(1)} \ldots z_{g_p}^{(p)}$, with $2 \leq p \leq 2n-1$, for each $g_1, \ldots, g_p \in G_0$ such that $x_{g_1}^{(1)} \ldots x_{g_p}^{(p)}$ is a graded identity of B.*

Remark 4.5 *It is interesting to observe that in characteristic p case the map ζ_J also maps multilinear identities of B into multilinear identities of $B \otimes E$. But such identities may not be enough to generate all $G \times \mathbb{Z}_2$-graded identities of $B \otimes E$, since in positive characteristic, the identities may not be generated by the multilinear ones.*

As an example one can consider the field F, as the algebraic closure of the prime field $\mathbb{Z}_p$, graded by the trivial group. The ideal of identities of F are generated by the polynomial $[x_1, x_2]$, but the algebra $F \otimes E$, which is isomorphic to E, satisfies the $\mathbb{Z}_2$-graded identity

$$St_p(y_1, \ldots, y_p) = \sum_{\sigma \in S_p} (-1)^\sigma y_{\sigma(1)} \cdots y_{\sigma(p)}$$

which is not in the $T_{\mathbb{Z}_2}$-ideal generated by the image of $[x_1, x_2]$ by ζ_J.

Problems involving relations between identities in positive characteristic and in characteristic zero are quite difficult. See for example [31, Problem e), p. 185].

5 Color Commutative Superalgebras

In this section we study the connection between identities of a G-graded algebra R and the identities of the tensor product of R by an H-graded color commutative superalgebra, C, where H is an abelian group. Again, we work over a field of characteristic zero.

If H is an abelian group, written additively, let $\beta : H \times H \longrightarrow F^*$ be a skew-symmetric bicharacter, i.e., a function satisfying for all $g, h, k \in H$, the following properties

$$\beta(g+h,k) = \beta(g,k)\beta(h,k)$$
$$\beta(g,h+k) = \beta(g,h)\beta(g,k)$$
$$\beta(g,h) = \beta(h,g)^{-1}$$

Now, if $C = \oplus_{h\in H} C_h$, we define the β-commutator

$$[a,b]_\beta = ab - \beta(h,k)ba$$

for $a \in C_h$ and $b \in C_k$ and extend it by linearity to C. We say that C is β-commutative if $[a,b]_\beta = 0$, for all a, $b \in C$. If β is fixed we call β-commutative algebras "color commutative superalgebras" [7].

As examples, if one considers $\beta \equiv 1$, β-commutative algebras are simply commutative algebras. If one considers the Grassmann algebra E, with its usual $\mathbb{Z}_2$-grading, one can see that defining $\beta(g,h) = 1$, if $g = 0$ or $h = 0$ and $\beta(g,h) = -1$ otherwise, one obtains that $[x,y]_\beta = 0$, for all x, $y \in E$, i.e., E is a color commutative superalgebra.

The main result of this section is a generalization of Theorem 4.2 above [17, Theorem 11], i.e., we want to replace E by an arbitrary H-graded color commutative superalgebra C in the above theorem.

If R is a G-graded algebra and C is a H-graded color commutative superalgebra, of course, $R \otimes C$ is a $G \times H$-graded algebra.

If G is a group and $\mathbf{g} = (g_1, \ldots, g_n) \in G^n$, we denote by $P_{\mathbf{g}}$, the subspace of $F\langle X|G\rangle$ generated by $\{x_{g_{\sigma(1)}\sigma(1)} \cdots x_{g_{\sigma(n)}\sigma(n)} \mid \sigma \in S_n\}$. The space of multilinear polynomials of degree n of $F\langle X \mid G\rangle$ is defined by $P_n^G = \bigoplus_{\mathbf{g}\in G^n} P_{\mathbf{g}}$. It is well known that if F has characteristic zero, the G-graded identities of a G-graded F-algebra A, follow from the ones in $P_{\mathbf{g}}$, for $\mathbf{g} \in G^n$ and $n \in \mathbb{N}$.

If $\mathbf{g} = (g_1, \ldots, g_n) \in G^n$ and $\mathbf{h} = (h_1, \ldots, h_n) \in H^n$, we denote $\mathbf{g} \times \mathbf{h} = ((g_1, h_1), \ldots, (g_n, h_n)) \in (G \times H)^n$.

For each sequence of elements of H, $\mathbf{h} = (h_i)_{i\in\mathbb{N}}$, we define a map $\varphi_{\mathbf{h}} : F\langle X|G\rangle \longrightarrow F\langle X|G\times H\rangle$ as the unique homomorphism of G-graded algebras satisfying $\varphi_{\mathbf{h}}(x_{g_i i}) = x_{(g_i,h_i)i}$. Here the G-grading on $F\langle X|G\times H\rangle$ is the one induced by $\deg_G(x_{(g,h)i}) = g$. If $\mathbf{g} = (g_1, \ldots, g_n) \in G^n$ we denote by the same symbol, $\mathbf{g}$, the sequence $(g_i)_{i\in\mathbb{N}}$ such that g_i is the neutral element of G, if $i > n$.

Now, for each multilinear monomial $m \in F\langle X|G \times H\rangle$, in the variables $x_{(g_{i_1},h_{i_1})i_1}, \ldots, x_{(g_{i_k},h_{i_k})i_k}$, with $i_1 < \cdots < i_k$, we may write $m = x_{(g_{i_{\sigma(1)}} h_{i_{\sigma(1)}})i_{\sigma(1)}} \cdots x_{(g_{i_{\sigma(k)}} h_{i_{\sigma(k)}})i_{\sigma(k)}}$, for some permutation σ.

If in the free H-graded color commutative superalgebra, we have

$$x_{h_{i_1}i_1} \cdots x_{h_{i_k}i_k} = \lambda_{\mathbf{h}\sigma} x_{h_{i_{\sigma(1)}}i_{\sigma(1)}} \cdots x_{h_{i_{\sigma(k)}}i_{\sigma(k)}},$$

with $\lambda_{\mathbf{h}\sigma} \in F^*$, we define $\zeta(m) = \lambda_{\mathbf{h}\sigma} m$.

Also, for each multilinear monomial $m \in P_n^G$, we define the map $\phi_{\mathbf{h}}$ on m as $\phi_{\mathbf{h}}(m) = \zeta(\varphi_h(m))$ and extend $\phi_{\mathbf{h}}$ to P_n^G by linearity.

Theorem 5.1 *[6, Theorem 3.1] Let C be an H-graded color commutative superalgebra, generating the variety of all H-graded color commutative superalgebras and let R be any G-graded algebra. If $f(x_{g_1 1}, \ldots, x_{g_n n})$ is a multilinear G-graded polynomial and $\mathbf{h} = (h_1, \ldots, h_n) \in H^n$, then $f(x_{g_1 1}, \ldots, x_{g_n n}) = 0$ is a graded polynomial identity for the G-graded algebra R if and only if $\phi_{\mathbf{h}}(f)(x_{(g_1,h_1)1}, \ldots, x_{(g_n,h_n)n}) = 0$ is a graded polynomial identity for the $(G \times H)$-graded algebra $R \otimes C$.*

Although the above result associates G-graded identities of R with $G\times H$-identities of $R \otimes C$, it is not enough for our purposes. We want to construct a basis for the $G \times H$-graded identities of $R \otimes C$ starting from a basis of the G-graded identities of R. In order to construct such basis, we need to generalize some lemmas used in the proof of Theorem 4.2 [17, Theorem 11], namely Lemmas 5, 9 and 10 of [17] to the case of tensor product by color commutative superalgebras. This follows below.

We remark that the set J and the map ζ_J in section 4 plays the same role of the sequence $\mathbf{h}$ and the map $\phi_{\mathbf{h}}$ in this section.

Lemma 5.2 *Let $\mathbf{g} \in G^n$ and $\mathbf{h} \in H^n$. If $f \in T_{G\times H}(R \otimes C) \cap P_{\mathbf{g}\times\mathbf{h}}$. Then there exists $f_0 \in P_{\mathbf{g}} \cap T_G(R)$ such that $f = \phi_{\mathbf{h}}(f_0)$.*

Proof. If $f \in T_{G\times H}(R\otimes C)\cap P_{\mathbf{g}\times\mathbf{h}}$ we may write

$$f=\sum_{\sigma\in S_n}\alpha_\sigma x_{(g_{\sigma(1)},h_{\sigma(1)})\sigma(1)}\cdots x_{(g_{\sigma(n)},h_{\sigma(n)})\sigma(n)},$$

for some $\alpha_\sigma\in F$. Now we define f_0 as

$$f_0=\sum_{\sigma\in S_n}\alpha_\sigma\lambda_{\mathbf{h}\sigma}^{-1}x_{g_{\sigma(1)}\sigma(1)}\cdots x_{g_{\sigma(n)}\sigma(n)},$$

where $\lambda_{\mathbf{h}\sigma}\in F^*$ is the coefficient used in the definition of $\phi_{\mathbf{h}}$.

Now it is easy to observe that $\phi_{\mathbf{h}}(f_0)=f$. By Theorem 5.1, $f_0\in T_G(R)$ if and only if $f\in T_{G\times H}(R\otimes C)$ and this proves the lemma. □

Lemma 5.3 *Let $u_1,\ \ldots,\ u_m$ be monomials in $F\langle X|G\rangle$ such that for some $n\geq m$, $\mathbf{g}\in G^n$ and $\mathbf{h}\in H^n$, $u=u_1\cdots u_m\in P_{\mathbf{g}\times\mathbf{h}}$. Consider $\mathbf{h}'=(h'_1,\ldots,h'_m)\in H^m$ such that $h'_i=\deg_H(u_i)$. Then, there exists $\gamma\in F^*$ such that for every $\sigma\in S_m$,*

$$\zeta(u_{\sigma(1)}\cdots u_{\sigma(m)})=\gamma\lambda_{\mathbf{h}'\sigma}u_{\sigma(1)}\cdots u_{\sigma(m)}$$

Proof. By the definition of the map ζ, there exist $\gamma\in F^*$ such that

$$\zeta(u_1\cdots u_m)=\gamma u_1\cdots u_m.$$

Since in the free H-graded color commutative superalgebra we have

$$\lambda_{\mathbf{h}'\sigma}x_{h'_{\sigma(1)}\sigma(1)}\cdots x_{h'_{\sigma(m)}\sigma(m)}=x_{h'_1 1}\cdots x_{h'_m m}$$

and for each $i\in\{1,\ldots,m\}$, $\deg(u_i)=h'_i=\deg(x_{h'_i i})$, we obtain that

$$\zeta(u_{\sigma(1)}\cdots u_{\sigma(m)})=\gamma\lambda_{\mathbf{h}'\sigma}u_{\sigma(1)}\cdots u_{\sigma(m)},$$

which proves the lemma. □

Lemma 5.4 *Let $f=f(x_{g_1 1},\ldots,x_{g_m m})\in P_m^G$, and $w_1,\ldots,w_m$ monomials in $F\langle X|G\rangle$ such that $\deg_G(w_i)=g_i$, for each i and $f(w_1,\ldots,w_m)\in P_n^G$. If $\mathbf{h}\in H^n$, there exist $\mathbf{h}'\in H^m$ and homogeneous elements $b_1,\ldots,b_m\in F\langle X|G\times H\rangle$ such that $\phi_{\mathbf{h}}(f(w_1,\ldots,w_m))=\gamma\phi_{\mathbf{h}'}(f)(b_1,\ldots,b_m)$*

Proof. For each $i\in\{1,\ldots,m\}$, define $b_i:=\varphi_{\mathbf{h}}(w_i)$ and $h'_i:=\deg_H(b_i)$ and let $\mathbf{h}':=(h'_1,\ldots,h'_m)\in H^m$.

Suppose now that

$$f(x_{g_1 1}, \ldots, x_{g_m m}) = \sum_{\sigma \in S_m} c_\sigma x_{g_{\sigma(1)}\sigma(1)} \cdots x_{g_{\sigma(m)}\sigma(m)}.$$

Then, if $\tilde{f} = \phi_{\mathbf{h}'}(f)$, we have

$$\tilde{f}(x_{(g_1,h'_1)1}, \ldots, x_{(g_m,h'_m)m}) = \sum_{\sigma \in S_m} c_\sigma \lambda_{\mathbf{h}'\sigma} x_{(g_{\sigma(1)},h'_{\sigma(1)})\sigma(1)} \cdots x_{(g_{\sigma(m)},h'_{\sigma(m)})\sigma(m)}.$$

Thus,

$$\tilde{f}(b_1, \ldots, b_m) = \sum_{\sigma \in S_m} c_\sigma \lambda_{\mathbf{h}'\sigma} b_{\sigma(1)} \cdots b_{\sigma(m)}.$$

On the other hand,

$$\begin{aligned}
\phi_{\mathbf{h}}(f(w_1, \ldots, w_m)) &= \phi_{\mathbf{h}}(\sum_{\sigma \in S_m} c_\sigma w_{\sigma(1)} \cdots w_{\sigma(m)}) \\
&= \sum_{\sigma \in S_m} c_\sigma \zeta(\varphi_{\mathbf{h}}(w_{\sigma(1)} \cdots w_{\sigma(m)})) \\
&= \sum_{\sigma \in S_m} c_\sigma \zeta(\varphi_{\mathbf{h}}(w_{\sigma(1)}) \cdots \varphi_{\mathbf{h}}(w_{\sigma(m)})) \\
&= \sum_{\sigma \in S_m} c_\sigma \zeta(b_{\sigma(1)} \cdots b_{\sigma(m)}) \\
&= \sum_{\sigma \in S_m} c_\sigma \gamma \lambda_{\mathbf{h}'\sigma} b_{\sigma(1)} \cdots b_{\sigma(m)}.
\end{aligned}$$

The last equality follows from Lemma 5.3. Now, comparing the equations the result follows. $\square$

Finally, we state the theorem which generalizes Theorem 4.2

Theorem 5.5 *Let C be an H-graded color commutative superalgebra, generating the variety of all H-graded color commutative superalgebras and let R be any G-graded algebra. If $\mathcal{E} \subseteq \bigcup_{\substack{\mathbf{g} \in G^n \\ n \in \mathbb{N}}} P_{\mathbf{g}}$ is a system of multilinear generators for $T_G(R)$, then the set*

$$S = \{\phi_{\mathbf{h}}(f) \mid f \in \mathcal{E},\, \mathbf{h} \in H^n, n \in \mathbb{N}\}$$

is a system of multilinear generators of $T_{G \times H}(R \otimes C)$.

Proof. Let U be the T-ideal generated by S in $F\langle X|G\times H\rangle$. Of course $U\subseteq T_{G\times H}(R\otimes C)$, by Theorem 5.1. Let us now suppose that $f\in T_{G\times H}(R\otimes C)$. Since the characteristic of K is zero, we may assume that $f\in P_{\mathbf{g}\times\mathbf{h}}$, for some $\mathbf{g}\in G^n$ and $\mathbf{h}\in H^n$. By Lemma 5.2, there exists $f_0\in T_G(R)\cap P_{\mathbf{g}}$ such that $f=\phi_{\mathbf{h}}(f_0)$. Since $f_0\in T_G(R)$, $f_0\in\langle\mathcal{E}\rangle$. Then, there exist $f_1,\ldots,f_n\in\mathcal{E}$, $u_i,v_i,w_j^i\in F\langle X|G\rangle$ monomials, and $\alpha_i\in F$ such that

$$f_0=\sum_i \alpha_i u_i f_i(w_1^i,\ldots,w_{k_i}^i)v_i$$

Since $f_0\in P_{\mathbf{g}}$, we may assume that u_i,v_i,w_j^i are also multilinear. On the other hand, for each i,

$$\phi_{\mathbf{h}}(u_i f_i(w_1^i,\ldots,w_{k_i}^i)v_i)=\beta_i\phi_{\mathbf{h}}(u_i)\phi_{\mathbf{h}}(f_i(w_1^i,\ldots,w_{k_i}^i))\phi_{\mathbf{h}}(v_i)$$

for some $\beta_i\in F^*$. Now Lemma 5.4 implies that there exist $\mathbf{h^i}\in H^{k_i}$ and homogeneous elements $b_j^i\in F\langle X|G\times H\rangle$, such that

$$\phi_{\mathbf{h}}(f_i(w_1^i,\ldots,w_{k_i}^i))=\gamma_i\phi_{\mathbf{h^i}}(f_i)(b_1^i,\ldots,b_{k_i}^i),$$

for some $\gamma_i\in F^*$. Hence, Theorem 5.1 implies that for each i, $\phi_{\mathbf{h^i}}(f_i(b_1^i,\ldots,b_{k_i}^i))\in U$, and then every summand of $\phi_{\mathbf{h}}(f_0)$ is in U. As a consequence, $f\in U$ and $T_{G\times H}(R\otimes C)\subseteq U$. $\square$

The above theorem has an interesting application. To show that we need to recall some results on classification of gradings on $M_n(F)$, when F is an algebraic closed field of characteristic zero. Below follows a result on the classification of abelian gradings on block-triangular matrices [37], which generalize the classification of abelian gradings on full matrix algebras [8].

Theorem 5.6 *Let G be an abelian group and let the field F be algebraically closed. For any G-grading of the matrix algebra $UT(d_1,\ldots,d_m)$ there exist integers t, $q_1,\ldots,q_m$ such that $d_i=tq_i$, for each i, a subgroup H of G and a q-tuple $\mathbf{g}=(g_1,\ldots,g_q)\in G^q$ ($q=q_1+\cdots q_m$), such that $UT(d_1,\ldots,d_m)$ is isomorphic to $M_t(F)\otimes UT(q_1,\ldots,q_m)$ as a G-graded algebra where $M_t(F)$ is an H-graded algebra with a fine H-grading and $UT(q_1,\ldots,q_m)$ has an elementary grading defined by $\mathbf{g}=(g_1,\ldots,g_q)$.*

Moreover, it turns out that such fine H-grading on $M_t(F)$ makes it an H-graded color commutative superalgebra as we can see in the next result.

Theorem 5.7 (Theorem 3.4 (i), [6]) *Let $M_t(F)$ have a an H-fine grading with all homogeneous components one-dimensional. Then the H-graded polynomial identities $[x_{h_1},x_{h_2}]_\beta=0$, where h_1, $h_2\in H$, form a basis of the graded polynomial identities of $M_t(F)$.*

The above means that $M_t(F)$ generates the variety of all H-graded β-commutative superalgebras. And now we obtain

Corollary 5.8 *Let F be an algebraically closed field of characteristic zero and G be a finite abelian group. If the algebra $UT(d_1, \dots, d_m)$ is G-graded, write $UT(d_1, \dots, d_m) \cong M_t(F) \otimes UT(q_1, \dots, q_m)$ where M_t has a fine H-grading (H a subgroup of G), $d_i = tp_i$, for each i and $UT(q_1, \dots, q_m)$ has an elementary grading induced by a q-tuple $(g_1, \dots, g_q)$ of elements of G, with $q = q_1 + \dots + q_m$.*

If $g_1, \dots, g_q$ are pairwise distinct, then the $G \times H$-graded polynomial identities of $UT(d_1, \dots, d_m)$, follows from the polynomials $\phi_{\mathbf{h}}(f)$, with $\mathbf{h} \in H^n$ and f in the identities of (3) – (5) and the G-graded monomial identities of degree up to $2q-1$ of $UT(q_1, \dots, q_m)$.

Proof. By Theorem 5.7, the fine H-graded algebra $M_t(F)$ is an H-graded color commutative superalgebra generating the variety of all H-graded color commutative superalgebras. By Theorem 3.7, a basis for the identities of $UT(q_1, \dots, q_m)$ consists of the identities (3) – (5) and a finite number of graded monomial identities of degree up to $2q-1$, with $q = q_1 + \dots + q_m$. So Theorem 5.5 implies that the above is a basis for $UT(d_1, \dots, d_m)$, since applying the map $\phi_{\mathbf{h}}$ does not change the degree of a monomial. □

To finish this paper, we study the relation between the notion of H-graded color commutative superalgebras and the notion of a Regular H-grading.

If H is a group, the notion of a regular H-grading was introduced by Regev and Seeman [33] and we recall it now.

If H is a group an H-grading on an associative algebra $A = \bigoplus_{h \in H} A_h$ is called a regular H-grading if it satisfies the following two conditions:
1) For any $(h_1, \dots, h_n) \in H_n$, there exist $a_{h_1} \in A_{h_1}, \dots, a_{h_n} \in A_{h_n}$ such that $a_{h_1} \cdots a_{h_n} \neq 0$.
2) If $g, h \in H$, there exist $\theta(g,h) \in F^*$ such that for any $a \in A_g$, and $b \in A_h$, $ab = \theta(g,h)ba$.

Observe that if H is an abelian group, the condition 2) means that A is an H-graded color commutative superalgebra. Indeed, we claim that map $\theta : H \times H \longrightarrow F^*$ defined above is a skew-symmetric bicharacter.

To show that, let $g, h \in H$ and consider $a \in A_g$ and $b \in A_h$ such that $ab \neq 0$. Then on one hand one has $ab = \theta(g,h)ba$, which implies that $ba = \theta(g,h)^{-1}ab$. On the other hand $ba = \theta(h,g)ab$. Since $ab \neq 0$, we conclude that $\theta(h,g) = \theta(g,h)^{-1}$.

Also, if g, h, and $k \in H$, consider $a \in A_g$, $b \in A_h$ and $c \in A_k$ such that $abc \neq 0$. On one hand we have $abc = \theta(g, h+k)bca$. On the other hand, $abc = \theta(g,h)bac = \theta(g,h)\theta(g,k)bca$. Again, since $abc \neq 0$, and A is associative, we obtain that $\theta(g, h+k) = \theta(g,h)\theta(g,k)$.

We have just proved the following result (which is also mentioned in [3, Section 3.2]):

Lemma 5.9 *If an H-grading on an associative algebra A is regular, then A is a color commutative superalgebra.*

Observe now that condition 1) above means that A does not satisfy any monomial identity, and by condition 2) any multilinear polynomial is equivalent to a monomial. As a consequence, A does not satisfies any other polynomial identity. In particular, the polynomial identities of A are all consequences of the ones obtained from condition 2), i.e., all such identities are consequence of the θ-commutator. Which is the same as saying that A generates the variety of H-graded color commutative superalgebras.

Conversely, if A is an H-graded color commutative superalgebra generating the variety of all H-graded color commutative superalgebras, one easily sees that condition 1) and 2) above are verified. The above proves the next result.

Theorem 5.10 *Let H be an abelian group and A be an associative H-graded algebra. Then the H-grading on A is regular if and only if A is an H-graded color commutative superalgebra generating the variety of all H-graded color commutative superalgebras.*

Acknowledgements

This work was completed while the first author was a postdoctoral fellow at Memorial University of Newfoundland. He would like to thank prof. Yuri Bahturin for useful discussions on this subject.

References

[1] E. Aljadeff, A. Kanel-Belov, *Representability and Specht problem for G-graded algebras*, Adv. Math., 225 (5), (2010), 2391–2428.

[2] E. Aljadeff, D. Ofir *On group gradings on PI-algebras*, J. Algebra 428 (2015), 403–424.

[3] E. Aljadeff, D. Ofir *On regular G-gradings*, Trans.; Amer. Math. Soc. (2015), in press.

[4] S. S. Azevedo, *Graded identities for the matrix algebra of order n over an infinite field*, Comm. Algebra 30 (2002), no. 12, 5849–5860.

[5] S. S. Azevedo, *A basis for $\mathbb{Z}$-graded identities of matrices over infinite fields*, Serdica Math. J. 29 (2003), no. 2, 149–158.

[6] Yu. Bahturin, V. Drensky, *Graded polynomial identities of matrices*, Linear Algebra and its Applications, 357 (2002), 15–34.

[7] Yu. A. Bahturin, A. A. Mikhalev, V. M. Petrogradsky, M. Z. Zaicev, *Infinite Dimensional Lie Superalgebras*, Walter De Gruyter, Berlin, 1992.

[8] Yu. A. Bahturin, S. K. Sehgal, M. V. Zaicev, *Group gradings of associative algebras*, J. Algebra 241 (2001) 677 – 698.

[9] Yu. Bahturin, M. V. Zaicev *Group gradings on matrix algebras*, Canad. Math. Bull. 45 (2002), no. 4, 499–508.

[10] M Bărăscu, S. Dăscălescu *Good gradings on upper block triangular matrix algebras*, Communications in Algebra, 41 (2013), 4290–4298.

[11] A. Ya. Belov, *On non-Specht varieties* (Russian), Fundam. Prikl. Mat. 5 (1999), No. 1, 47–66.

[12] L. Centrone, T. C. de Mello, *On $\mathbb{Z}_n$-graded identities of block-triangular matrices*, Linear and Multilinear Algebra 63 (2015), no. 2, 302–313.

[13] O. M. Di Vincenzo, *On the graded identities of $M_{1,1}(E)$*, Israel J. Math. 80 (1992), no.3, 323–335.

[14] O. M. Di Vincenzo, R. La Scala, *Block-triangular matrix algebras and factorable ideals of graded polynomial identities.*

[15] O. M. Di Vincenzo, V. Nardozza, *Graded polynomial identities of verbally prime algebras*, Journal of Algebra and its Applications, Vol. 6, No. 3 (2007), 385–401.

[16] O. M. Di Vincenzo, V. Nardozza, $\mathbb{Z}_{k+l}\times\mathbb{Z}_2$ *identities for* $M_{k,l}\otimes E$, Rend. Sem. Mat. Univ. Padova, Vol. 108 (2002)

[17] O. M. Di Vincenzo, V. Nardozza, *Graded polynomial identities for tensor products by the Grassmann Algebra*, Comm. Algebra, Vol. 31 (2003), no. 3, 1453–1474.

[18] O. M. Di Vincenzo, P. Koshlukov, A. Valenti*Gradings on the algebra of upper triangular matrices and their graded identities*, Journal of Algebra 275 (2004) 550–566.

[19] O. M. Di Vincenzo, E. Spinelli *Graded polynomial identities on upper block triangular matrix algebras*, Journal of Algebra 415 (2014), 50–64.

[20] D. Diniz, *On the graded identities for elementary gradings in matrix algebras over infinite fields*, Linear Algebra and its Applications 439 (2013), 1530–1537.

[21] V. Drensky, *Free algebras and PI-algebras: Graduate Course in Algebra*, Springer, Singapore, (1999).

[22] V. Drensky, *A minimal basis for a second-order matrix algebra over a field of characteristic 0*, Algebra i Logika 20, no.3 (1980), 282–290 [in Russian]; Algebra and Logic 20, no. 3 (1981), 188–194 [Engl. transl.].

[23] A. Giambruno, M. Zaicev, *Minimal varieties of algebras of exponential growth*, Adv. Math. 174 (2003), no. 2, 310–323.

[24] A. Giambruno, M. Zaicev, *Polynomial Identities and Asymptotic Methods*, Mathematical Surveys and Monographs, Volume 122 (2005).

[25] A. V. Grishin, *Examples of T-spaces and T-ideals over a field of characteristic 2 without the finite basis property* (Russian, Engligh summary), Fundam. Prikl. Mat. 5 (1999), No. 1, 101–118.

[26] A. Kemer, *Solution of the problem as to whether associative algebras have a finite basis of identities*, Dokl. Akad. Nauk SSSR 298 (1988), 273–277; translation in Soviet Math. Dokl. 37 (1988), 60–64.

[27] A. Kemer, *Ideals of Identities of Associative Algebras*, Translations of Monographs, vol. 87, Amer. Math. Soc., Providence, RI, 1991.

[28] P. Koshlukov, *Basis of the identities of the matrix algebra of order two over a field of characteristic* $p \neq 2$, J. Algebra 241 (2001), 410–434.

[29] P. Koshlukov, S. S. Azevedo, *Graded identities for* T*-prime algebras over fields of positive characteristic*, Israel J. Math. 128 (2002), 157–176.

[30] Yu. N. Maltsev, *A basis for the identities of the algebra of upper triangular matrices* (Russian), Algebra i Logika 10 (1971), 393–400. Translation: Algebra and Logic 10 (1971), 242–247.

[31] C. Procesi, *Rings with Polynomial Identities*, Dekker, New York, 1973.

[32] Yu. P. Razmyslov, *Finite basing of the identities of a matrix algebra of second order over a field of characteristic zero*, Algebra i Logika 12, no. 1 (1973), 83–113 [in Russian]; Algebra and Logic 12 (1973), 47–63 [Engl. transl.].

[33] A. Regev and T. Seeman, $\mathbb{Z}_2$-graded tensor products of p.i. algebras, J. Algebra 291 (2005), no. 1, 274?296.

[34] V. V. Shchigolev, *Examples of infinitely based T-ideals* (Russian, English summary), Fundam. Prikl. Mat. 5 (1999), N0. 1, 307–312.

[35] I. Sviridova, *Identities of pi-algebras graded by a finite abelian group*, Comm. Algebra 39 (2011), no. 9, 3462–3490.

[36] A. Valenti, M. V. Zaicev, *Group gradings on upper triangular matrices* Arch. Math. 89 (2007), 33–40.

[37] A. Valenti, M. V. Zaicev, *Abelian gradings on upper block triangular matrices* Canad. Math. Bull. 55 (2012), no. 1, 208–213.

[38] S. Yu. Vasilovsky $\mathbb{Z}$-*graded polynomial identities of the full matrix algebra* Comm. Algebra 26 (1998), no. 2, 601–612.

[39] S. Yu. Vasilovsky $\mathbb{Z}_n$-*graded polynomial identities of the full matrix algebra of order* n Proc. Amer. Math. Soc. 127 (1999), no. 12, 3517–3524.